Entangled Photon Sources

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Quantum Networking Hardware: Entangled Photon Sources

Summary

Entangled photon sources are a critical component for entanglement-based quantum networks, which can provide secure communication and a variety of other capabilities. This white paper presents a comprehensive overview of foundational concepts and advanced techniques in entangled photon generation.

Introduction

Entangled photon states are essential to entanglement-based quantum networks. These entangled photon states can be distributed across nodes to generate information-theoretically secure keys, as well as other use cases such as networking quantum processing units to increase computational power and networking quantum sensors for precision measurement not possible through classical methods. This paper focuses on methods for generating entangled photons using nonlinear optical processes such as spontaneous parametric down-conversion (SPDC) and spontaneous four-wave mixing (SFWM).

To generate entanglement, photons from a laser are pumped into a nonlinear medium, such as a bulk crystal or a waveguide. In the case of SPDC, inside the nonlinear medium, a small fraction of the pump photons each split into two lower-energy entangled photons, known as a signal photon and an idler photon. In SFWM, two pump photons interact with the medium to generate two entangled signal and idler photons. Throughout this white paper, the photons in these processes are defined as:

- Pump photons: the photons introduced into the nonlinear crystal.
- Signal and idler photons: the pair of entangled photons generated by these nonlinear processes.

Entanglement is a non-classical correlation between two photons, such that their combined state cannot be represented independently. In other words, when two photons are entangled, they are intrinsically linked, or interdependent, and described by a non-separable quantum state. This property is central to the unique capabilities that entangled photons enable in applications like quantum networking, quantum computing and metrology.

Physics of Entangled Photon Sources

Understanding how entangled photons are created requires understanding a few foundational physics concepts that underpin the nonlinear optical processes involved in creating them. These processes rely on the rules of conservation of energy, the conservation of momentum, and the nonlinear effects that facilitate entanglement generation.

How do photons become entangled?

Entanglement can occur through processes like Spontaneous Parametric Downconversion or Spontaneous Four-Wave Mixing.

They use nonlinear optics.

Spontaneous Parametric Downconversion (SPDC)

Under special conditions, a single photon can spontaneously transform into two new photons, with their properties that are strongly and non classically correlated.

Spontaneous Four-Wave Mixing (SFWM)

Similar to SPDC, but starts with two photons, leading to the spontaneous creation of two new photons.

The photons involved in these processes are categorized as:

- **Pump photons**: The initial photon(s) seeded into the nonlinear medium.
- **Signal photon**: One of the photons produced from the nonlinear process
- **Idler photon**: One of the photons produced from the nonlinear process

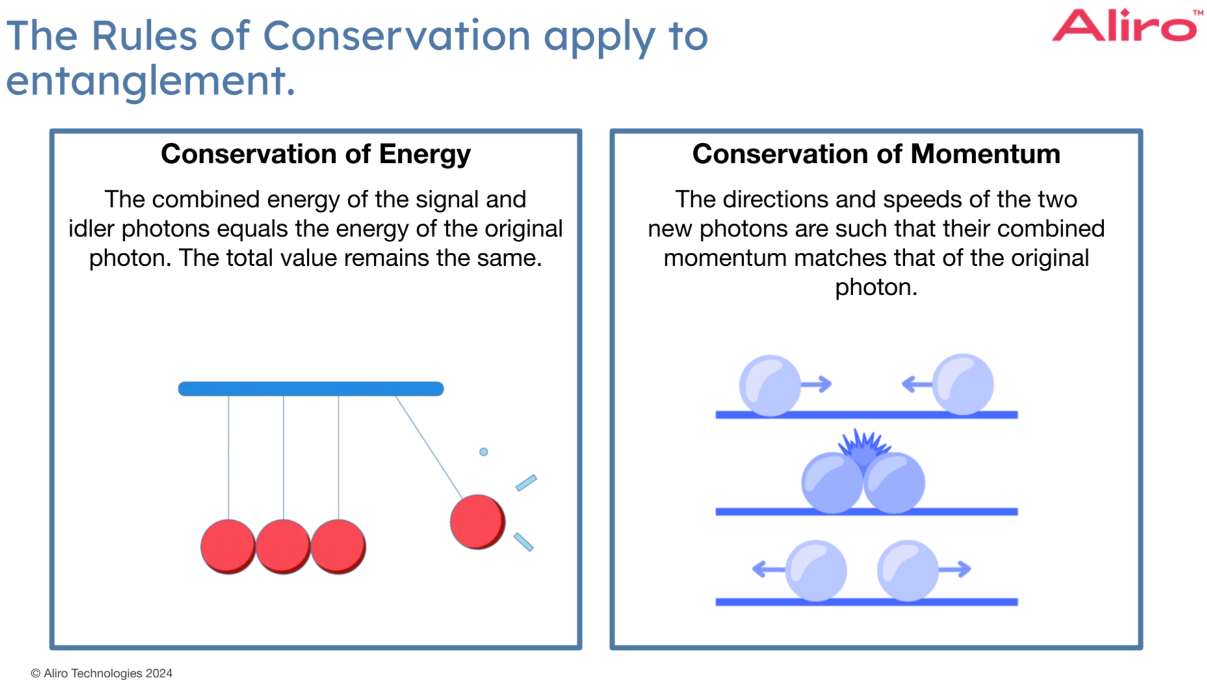
Conservation Laws and Nonlinear Optical Processes

Conservation of Energy:

One of the primary principles in generating entangled photons is the conservation of energy. In the processes discussed here, the energy of the output entangled photons is equal to the energy of the input pump photon/s responsible for their generation. This is characteristic of what are known as parametric processes, where energy is not transferred from the optical fields into the nonlinear medium itself. In other words, the medium's quantum state remains unaffected, and energy is conserved within the interacting optical fields.

Conservation of Momentum:

Another fundamental principle is conservation of momentum. This rule dictates that the combined momentum of the output entangled photons must match the momentum of the original input photon/s responsible for their generation. The directions and speeds of the output photons, when combined, reflect the momentum of the input photon.



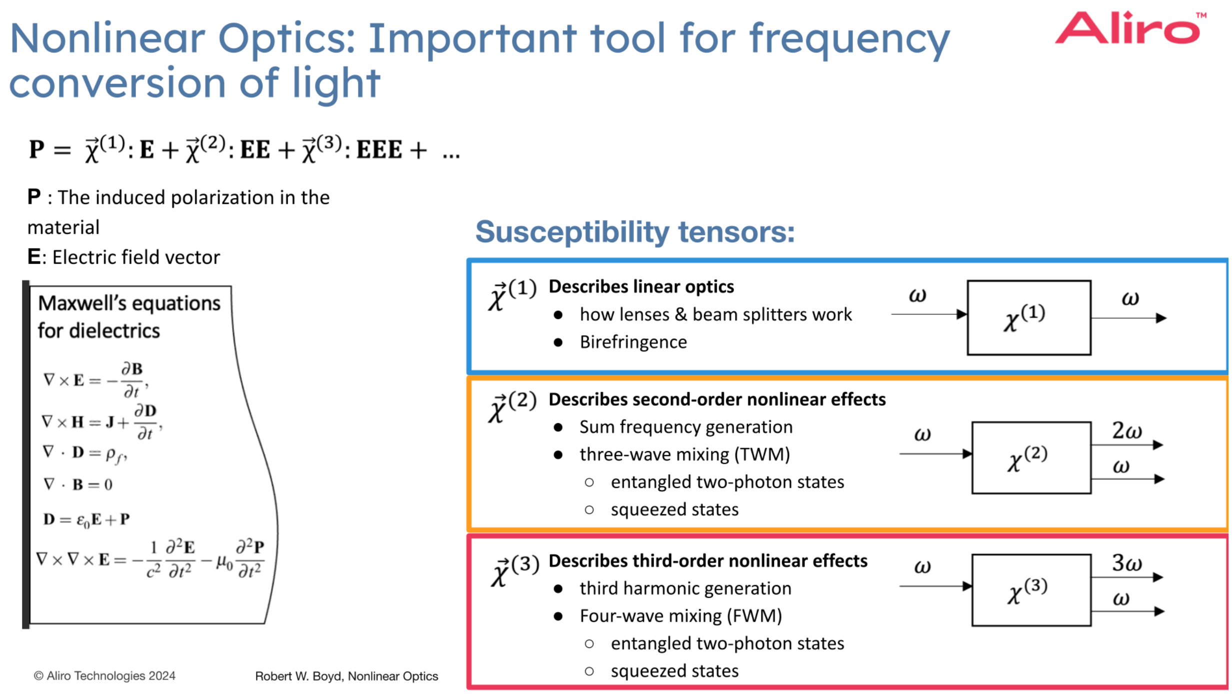
Nonlinear Optical Processes and Electron Dynamics

To understand nonlinear processes in detail, it's helpful to understand how bound electrons in the atoms of a medium interact with a strong electric field. We can use an analogy to illustrate this: imagine a mass on a spring.

In the linear regime, when a small displacement is applied to the mass on the spring, a force proportional to this displacement returns it to its equilibrium position. Such a linear response of the bound electrons to a weak electromagnetic field incident on the medium, explains many familiar optical behaviors, such as how lenses and beam splitters operate and how birefringence works. In this simple case, the input and output frequencies of the electric field are the same.

In the nonlinear regime, if the displacement is increased, it's no longer possible to describe the force using only a linear term. Higher-order terms come into play—quadratic, cubic, and beyond—which add complexity to the system's behavior. In the example of a mass on a spring, this nonlinear response results in a "jerky" anharmonic motion. This is analogous to the behavior of electrons in a medium when exposed to a strong electric field. The higher-order nonlinear terms of electric field become significant in describing the response of the electrons and introduce additional frequency components, which are essential for the nonlinear optical processes used in entangled photon generation.

To understand this nonlinear response quantitatively, consider the induced polarization in the material due to the electric fields that are present in the system.

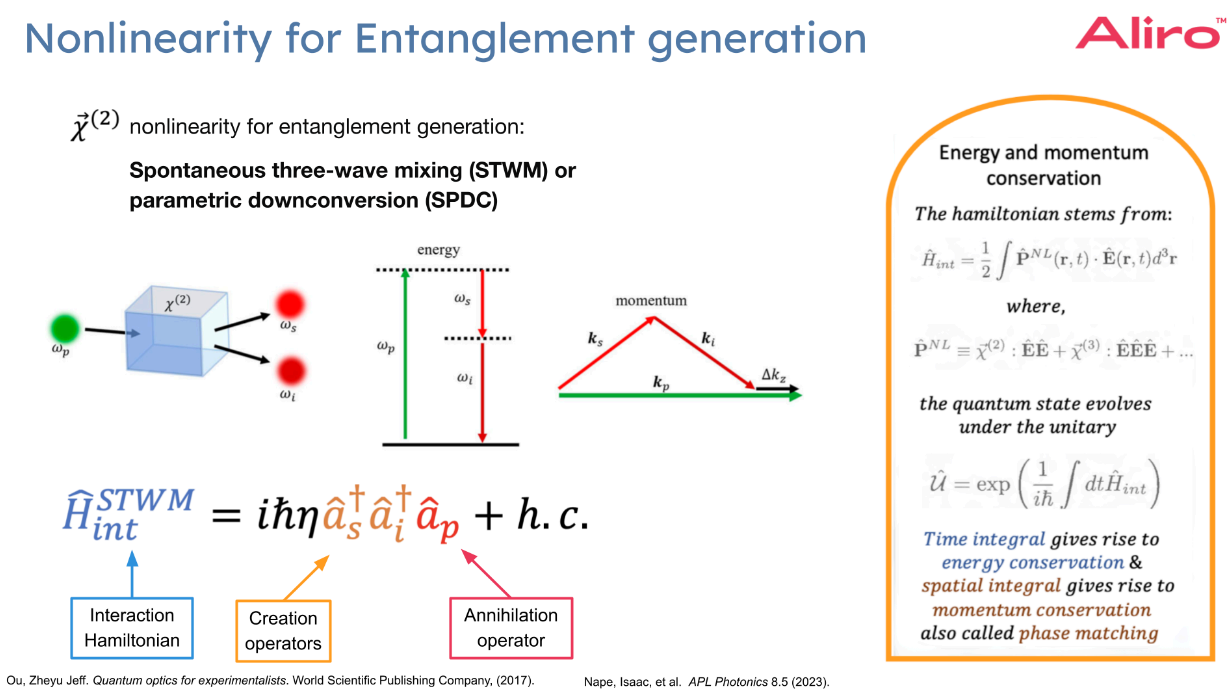


The polarization, represented as P, can be tied into Maxwell's equations for dielectrics, as shown in the left hand side in the image above. There are various terms that contribute to the polarization induced in a material as a function of the electric field in the material.[Boyd]

First, consider the linear term. This particular term dependent on the first-order susceptibility tensor, $\chi^{(1)}$, is going to determine essentially all the basic concepts that we are most familiar with: how lenses and beam splitters work, as well as how birefringence occurs. The key takeaway is that the input and output electric fields maintain the same frequency.

The second-order nonlinear term is described by the second-order susceptibility tensor, $\chi^{(2)}$, where interaction between three optical fields is captured: one field due to the polarization itself, and then another two which can be seen on the right hand side of the equation. The second-order term enables a variety of nonlinear optical processes driven by the interaction of three optical fields. One example is sum-frequency generation, where the output electric field's frequency is the sum of the frequencies of the input fields. Second-harmonic generation is a specific case where the output electric field has twice the frequency of the input field. Additionally, processes like three-wave mixing, which arise from this term, are instrumental in generating entangled photon states.

In three-wave mixing, or spontaneous parametric down-conversion, a high-frequency photon is introduced into a nonlinear medium. The output photons generated through the nonlinear process have energies (or frequencies) that sum to match the energy (or frequency) of the input photon, ensuring conservation of energy. Similarly, conservation of momentum is also ensured in this process, as detailed later.



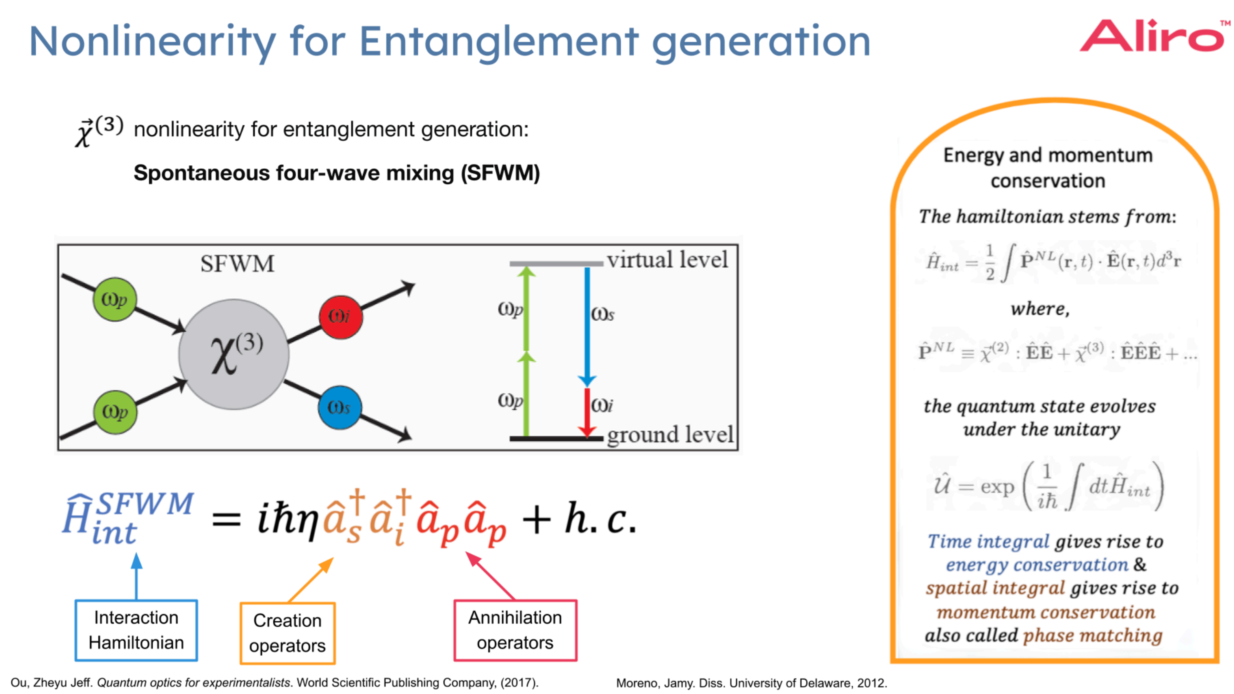
To understand from the quantum mechanical perspective how this comes about, it's helpful to take a closer look at the interaction energy of the optical fields that are involved–which is given by the interaction Hamiltonian, highlighted in blue in the image above.[Jeff Ou] This can be expressed as an integral over space of the dot product between the nonlinear term/s that strongly contribute to the polarization induced in the material (as described earlier) and the electric field present in the medium.

Using this Hamiltonian, the unitary evolution operator, as shown on the right-hand side of the image above, determines the evolution of the quantum state of the interacting fields. As the fields propagate through time and the length of the crystal, two integrals emerge. The first is an integral over space, appearing in the expression for the Hamiltonian. This integral gives rise to momentum conservation. The second is an integral over time in the expression for the unitary operator, which gives rise to energy conservation. The expression for the interaction Hamiltonian in the case of spontaneous parametric downconversion is shown in the bottom-left corner of the image above.

This Hamiltonian includes creation operators, representing the generation of a signal photon and an idler photon. It also includes an annihilation operator for the pump, representing the consumption of one pump photon in the process. However, it is important to note that when this Hamiltonian is used in the unitary evolution operator shown on the right-hand side, there will be terms arising where multiple signal and idler photons can also be generated and multiple pump photons can also be annihilated.

In this white paper, we will specifically focus on the fundamental term that describes the generation of two output photons and the consumption of one input pump photon.

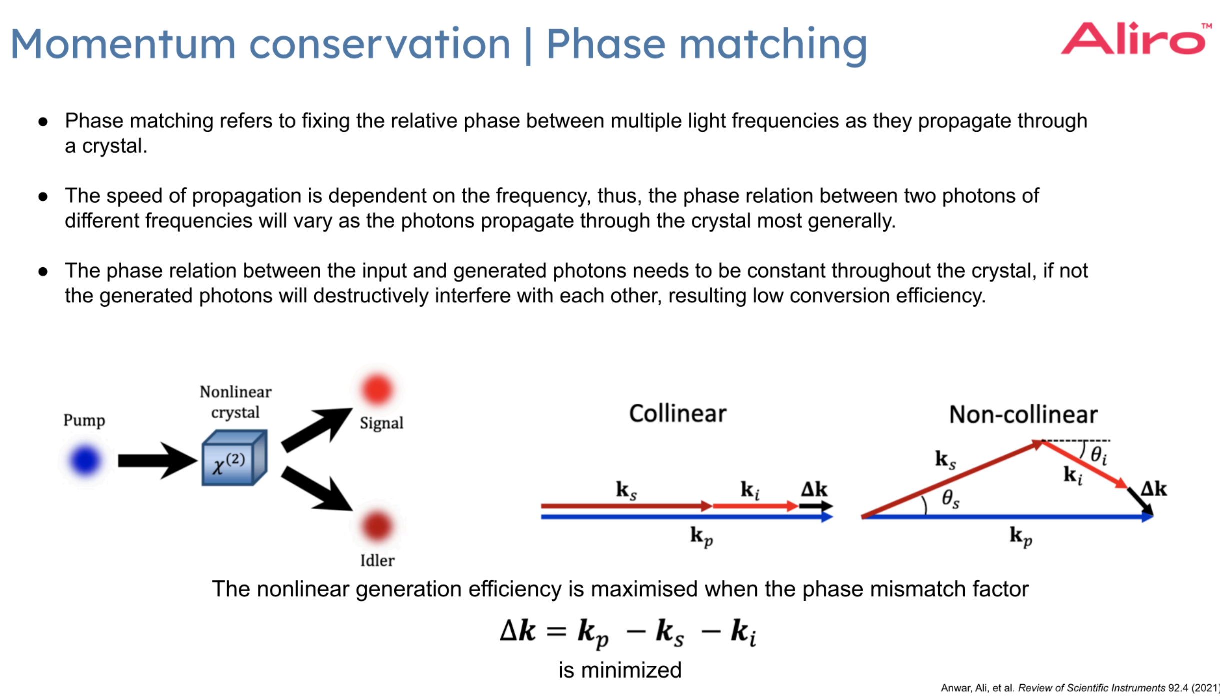
Furthermore, the third-order nonlinear effects, described by the third-order susceptibility tensor $\chi^{(3)}$, can describe nonlinear processes such as third harmonic generation, where the output electric field has three times the frequency of the input field. The third-order nonlinear term is also responsible for a process called four-wave mixing (FWM), which can be harnessed to generate entangled photon states, as detailed below.



In the case of the third-order nonlinear process of spontaneous four-wave mixing (SFWM), two pump photons are consumed to generate two entangled output photons: the signal photon and the idler photon. This process is described by the interaction Hamiltonian, shown in blue at the bottom of the image above.[Jeff Ou] The interaction Hamiltonian includes creation operators for the signal and idler photons, representing their generation, as well as two annihilation operators for pump photons, representing their consumption.

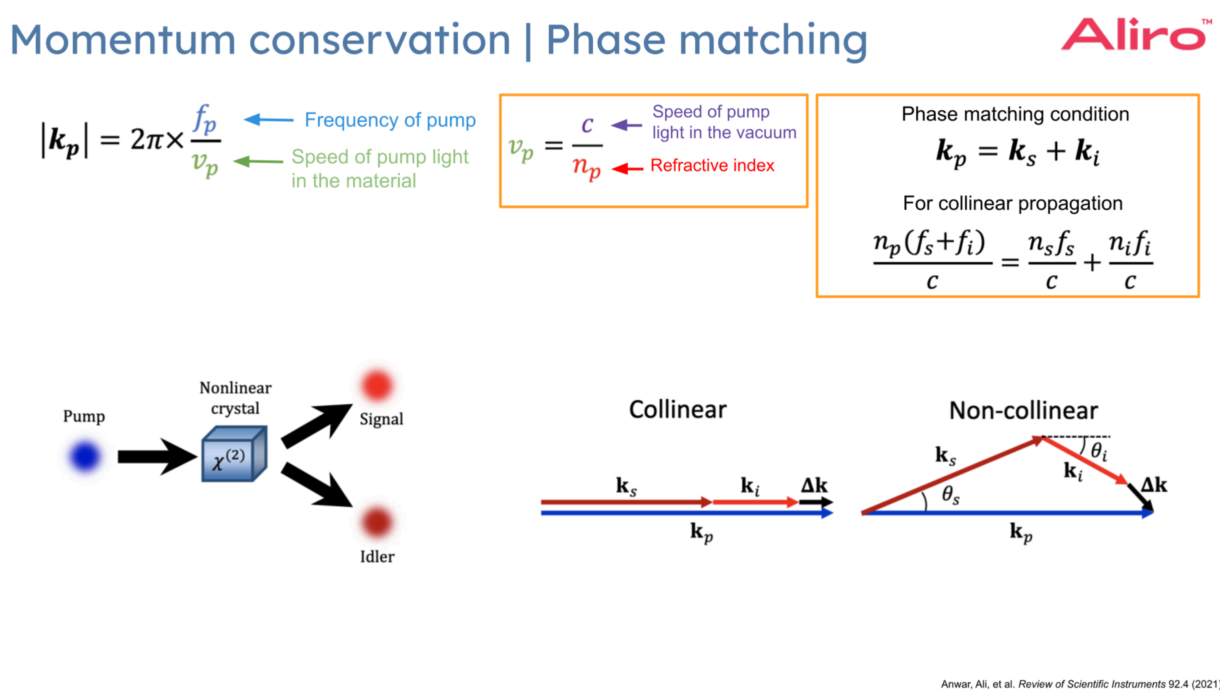
In both SPDC and SFWM, conservation of momentum is an important criterion in efficiently generating these entangled photons.

It's helpful to understand conservation of momentum further through the process of phase matching. Phase matching refers to maintaining a fixed relative phase between the output and the input electric fields throughout the length of the crystal. As these fields propagate through the crystal, their speeds, or phase velocities, differ due to the material's dispersive properties, that is, the speed of photons varies with their frequency. To maintain fixed relative phases along the crystal, specific phase-matching schemes must be engineered. Without phase matching, the efficiency of conversion from the input pump photons to the output generated photons will be reduced. Thus, phase matching is essential to realize high entanglement generation rates.



Phase-matching schemes must be devised based on the relative propagation of the fields. In collinear propagation, the pump, signal, and idler fields all travel in the same direction, such as in propagation through optical waveguides.

In non-collinear propagation, the pump, signal, and idler fields can propagate in different directions.[Anwar, Ali, et al.] In both scenarios, suitable phase-matching schemes must be employed to minimize the phase mismatch term shown in the image above.[Boyd] This term is essentially the difference between the pump wave vector and the combined wave vector of the signal and idler. When the phase mismatch is minimized and brought to zero, maximum efficiency is achieved in converting pump photons into output photons.



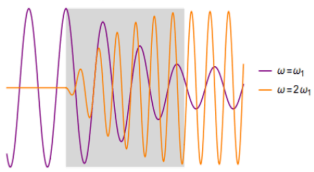
A close look at these wave vectors, or momentum vectors, reveals that their magnitudes can be expressed in terms of the photon's frequency and its speed in the material, as shown in the image above. Specifically, the speed of the photon in a material is less than the speed of the photon in vacuum. The factor that connects these is known as refractive index. For a specific case of collinear propagation, the phase-matching condition can be expressed as shown in the image above relating the frequencies and refractive indices of the signal, idler, and pump photons.

To visualize this, let's consider the example of second harmonic generation. Here, the output electric field generated from the non linear process has twice the frequency of the input electric field.

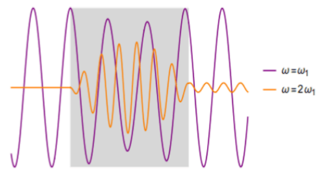
Phase matching: Visualization

Consider Second harmonic generation in a second order nonlinear crystal

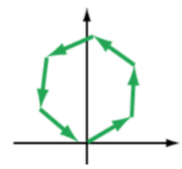
With phase matching



Without phase matching



Considering the addition of nonlinear process output amplitude contributions from different parts of the crystal, only with phase matching, a high conversion efficiency can be achieved.



https://en.wikipedia.org/wiki/File:Frequency_doubling_with_perfect_pase_matching.gif
https://en.wikipedia.org/wiki/File:Frequency_doubling_with_imperfect_phase_matching.gif

Consider the scenario where phase matching is satisfied (shown in the top-left figure of the image above). The second harmonic output (denoted by the orange waveform) increases throughout the length of the crystal, while the input pump (denoted by the violet waveform) decreases. In this case, the relative positions of the peaks and troughs of the input pump remain fixed with respect to those of the second harmonic output. In other words, the relative phase between the input and output fields is maintained throughout the length of the crystal. As a result, energy is efficiently transferred from the pump field to the output field along the entire crystal length.

However, if phase matching is not achieved, as shown in the top-right figure, the energy transferred from the pump to the second harmonic output during propagation can be transferred back to the pump at different points in the crystal. This results in poor overall conversion efficiency.

The bottom two figures, with green arrows, illustrate this concept in terms of the contributions to the second harmonic output along the crystal. When phase matching is satisfied, these contributions constructively interfere, leading to significant conversion efficiency. Conversely, without phase matching, the contributions destructively interfere, resulting in low output power.

We next pose the question: What schemes can be devised to achieve phase matching in a nonlinear medium?

Phase matching: Techniques

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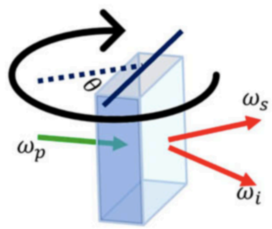
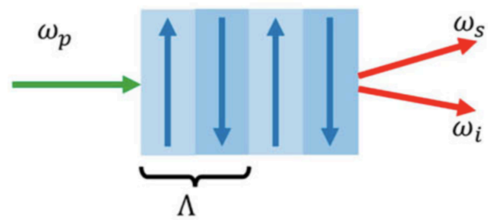
Birefringent phase matching



Light is propagated through the crystal in a direction where the natural birefringence of the crystal has the same refractive index as the generated photons

Quasi-phase matching



Orientation of the crystal axis can be engineered to achieve phase matching.

Nape, Isaac, et al. *APL Photonics* 8.5 (2023).

Two well-known techniques for achieving phase matching are birefringent phase matching and quasi-phase matching.

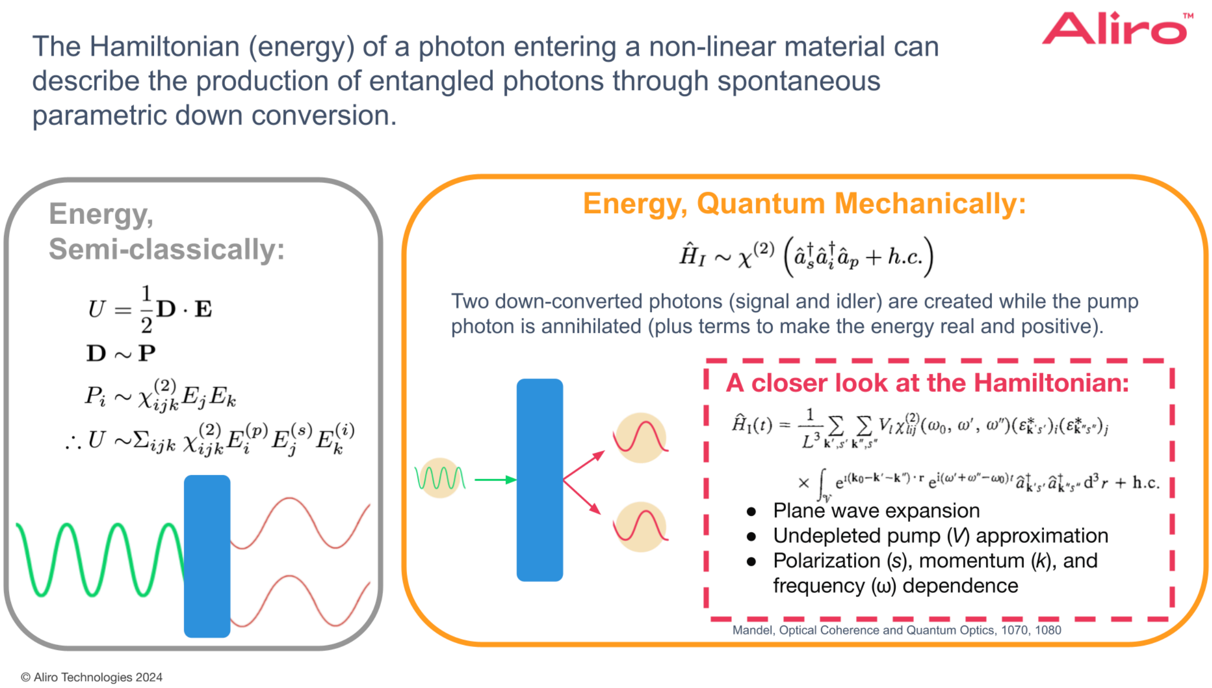
1. Birefringent Phase Matching: This technique is primarily used in non-collinear propagation geometries and leverages the material's birefringence, where photons with different polarizations travel at different speeds. By carefully aligning the frequencies, polarizations, and orientations of the optical fields propagating through the medium, phase matching can be achieved.
2. Quasi-Phase Matching: This approach involves engineering the crystal's axes periodically along its length to compensate for phase mismatches, ensuring effective phase matching over the propagation distance. Quasi-phase matching can be applied to both collinear and non-collinear propagation geometries.

Spontaneous Parametric Down Conversion in bulk crystals

Overview of SPDC

Spontaneous parametric down conversion occurs when one high-energy photon (a pump photon) interacts with a nonlinear material (such as a bulk crystal) and splits into two lower-energy (or down converted) entangled photons, known as the signal photon and idler photon. The process is spontaneous, meaning it requires no additional fields to stimulate it. The process is also a parametric process, meaning it preserves both energy and momentum between the input photon and output photons.

The Hamiltonian (a mathematical operator which returns the energy) of a photon entering a non-linear material shed insight on the process of spontaneous parametric down conversion.



The Hamiltonian can be represented semi-classically first to get an intuitive feel for what's happening. In a semi-classical picture, we can view light purely as electromagnetic waves and ignore the “particle” nature of light. In this picture, a higher energy wave that enters into a nonlinear material has some chance of becoming two lower energy waves. The energy associated with electromagnetic light within a material will depend on the displacement field D and the electric field E. The displacement field contains a term equal to the polarization. The polarization, as explained above, will contain many terms: terms that go by the $\chi^{(1)}$ birefringence term, $\chi^{(2)}$ nonlinearity, $\chi^{(3)}$ nonlinearity, and so on.

SPDC processes occur in materials with relatively large $\chi^{(2)}$ nonlinearities, so for SPDC, we only consider contribution from the $\chi^{(2)}$ term. The polarization will contain a term equal to the $\chi^{(2)}$nonlinearity multiplied by two electric fields. Intuitively, the two electric fields can be thought of as the electric field of the signal photon and the electric field of the idler photon. Plug in all these equations and the result looks something like the $\chi^{(2)\,)}$ nonlinearity multiplied by the electric fields associated with the pump photon, signal photon, and idler photon.

We live in a world where light is not purely classical: light can be also quantized as single photons of light that can act more like a particle, and sometimes can behave more like a wave. Treating this more quantitatively requires moving to a quantum mechanical picture. The interaction Hamiltonian in this quantum mechanical view clarifies nonlinear interaction responsible for generating the signal and idler photons.

It can be written as: $H_{interaction} \sim \chi^{(2)} (a^\dagger_{signal}\, a^\dagger_{idler}\, a_{pump} + h.c.)$

The full Hamiltonian for SPDC reveals several key properties that directly influence the nature of the entangled photon pairs produced. Within the Hamiltonian, we find terms representing:

- Phase Matching and Energy Conservation. These appear as plane wave terms, such as $e^{i\Delta k \cdot r}$ and $e^{i\Delta\omega \cdot t}$, where Δk and Δω represent momentum and energy differences, respectively. After integrating, these terms become phase matching and energy conservation terms.
- Frequency Dependence. The $\chi^{(2)}$ nonlinearity depends on the frequencies of the pump, signal, and idler photons (ω_0, ω' and ω'' in the equation above), allowing for fine-tuning of the process to produce photons at specific wavelengths.
  Undepleted Pump Approximation. The SPDC process often will assume an undepleted pump approximation, where the pump photon count is so large that the annihilation of pump photons is treated as negligible, approximating the pump photons as a constant field (**V in the equation above**). This simplifies the Hamiltonian.
- Polarization, Momentum, and Frequency Entanglement. The Hamiltonian contains dependence on polarization, momentum, and frequency, all of which contribute to the entangled nature of the output photons. These dependencies are fundamental to creating different types of entanglement (polarization, frequency, and momentum) between the signal and idler photons.

Now that the Hamiltonian has been defined, the next step is to solve for the wave functions. Wave functions describe the quantum state of the entangled photons.

Solving the Schrödinger Equation for the Interaction Hamiltonian

To find the wave function that describes the entangled photon pairs, we start by applying the Schrödinger equation:

$\psi(t)=\exp(-\, i/\hbar \int H_{interaction} dt)\, \psi(0)$

In this expression, ψ(t) represents the wave function at time t, evolving from an initial state ψ(0). The exponential term includes an integral of the interaction Hamiltonian over time, which governs the evolution of the system in response to the SPDC process. The interaction Hamiltonian incorporates the second-order nonlinearity ($\chi^{(2)}$), multiplied by creation and annihilation operators for the signal, idler, and pump photons, plus the Hermitian conjugate to ensure energy conservation.

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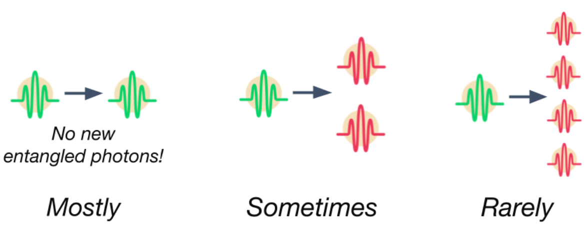
Solving the Schrödinger equation gives the wave function for the down-converted photons. Usually, no down-converted photons will be produced. Sometimes, a pair of down-converted photons will be produced. Rarely, multi-photon down-converted pairs will be produced.

Finding the wave function:

$$\hat{H}_I \sim \chi^{(2)} \left(\hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_p + h.c. \right)$$

$$|\Psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \int_0^t \hat{H}_I(t')dt' \right) |\Psi(0)\rangle$$

$$\approx \left(1 + \frac{-i}{\hbar} \int_0^t \hat{H}_I(t')dt' + \frac{1}{2} \left[\frac{-i}{\hbar} \int_0^t \hat{H}_I(t')dt' \right]^2 + ... \right) |\text{vac}\rangle$$



A closer look at the wave function:

$$|\Psi(t)\rangle = |\text{vac}\rangle_s |\text{vac}\rangle_i + L^{-3} \frac{1}{i\hbar} \sum_{[\mathbf{k}'s']_s} \sum_{[\mathbf{k}''s'']_i} V_l \chi_{lij}^{(2)}(\omega_0, \omega', \omega'')(\boldsymbol{\varepsilon}^*_{\mathbf{k}'s'})_i (\boldsymbol{\varepsilon}^*_{\mathbf{k}''s''})_j$$
$$\times \prod_{m=1}^{3} \left[\frac{\sin \frac{1}{2}(\mathbf{k}_0 - \mathbf{k}' - \mathbf{k}'')_m l_m}{\frac{1}{2}(\mathbf{k}_0 - \mathbf{k}' - \mathbf{k}'')_m} \right] e^{i(\omega' + \omega'' - \omega_0)t/2} \frac{\sin \frac{1}{2}(\omega' + \omega'' - \omega_0)t}{\frac{1}{2}(\omega' + \omega'' - \omega_0)} |\mathbf{k}', s'\rangle_s |\mathbf{k}'', s''\rangle_i$$
$$+ \ldots \qquad (22.4\text{–}20)$$

- Multiply over *m=3* orthogonal directions in the crystal
- Sum over polarization (s) and momentum (k)
- A phase-matching sinc comes from momentum conservation
- In the limit of t ≫ 1/ω, the sinc of ω becomes a Dirac delta function
- Eigenstates are entangled in momentum (k) and polarization (s)
- Full eigenstate is a squeezed state

Mandel, Optical Coherence and Quantum Optics, 1080

To simplify the solution, we expand the exponential in a Taylor series under the assumption that the interaction term, and therefore $\chi^{(2)}$ nonlinearity is small. This approach allows us to truncate the series after a few terms.

The truncated expansion reveals key components of the wave function:

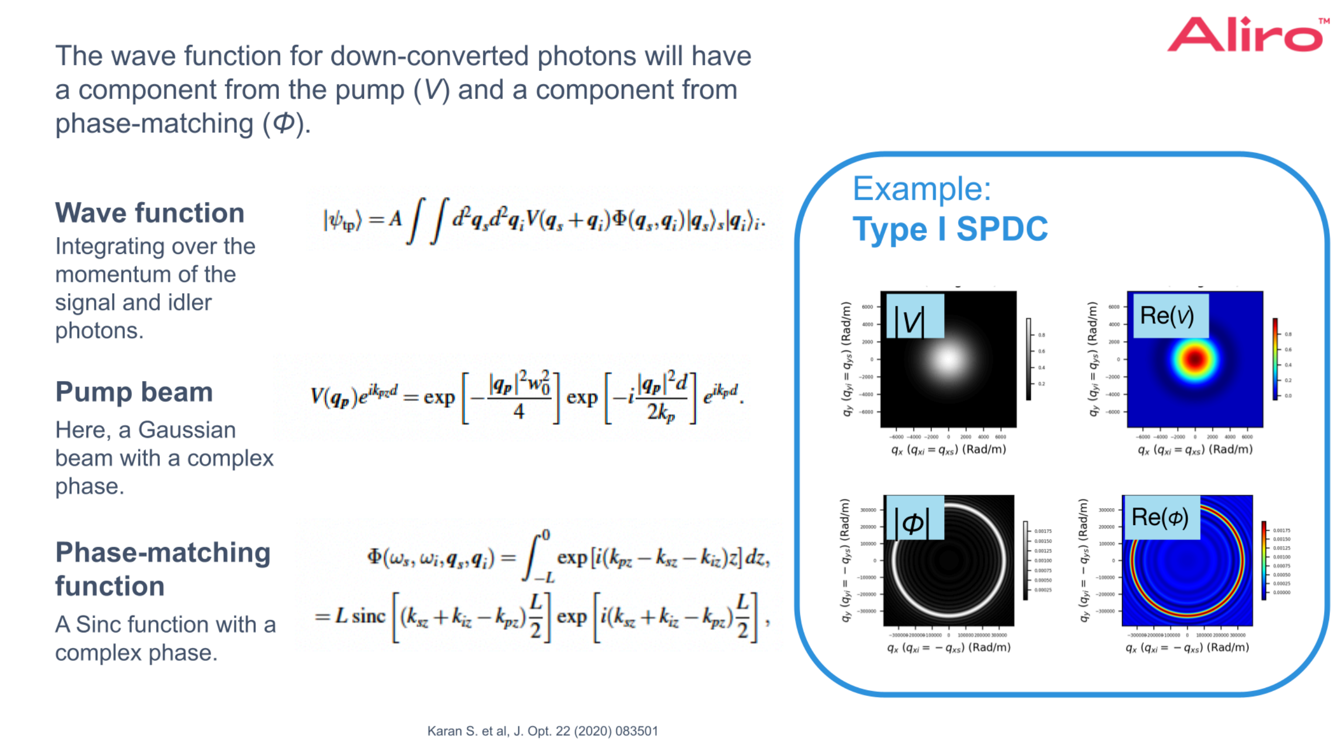
- **Vacuum State.** In most cases, no new entangled photon pairs are generated, and the system remains in the vacuum state.
- **Single Photon Pair Generation.** Sometimes the interaction Hamiltonian generates one signal photon and one idler photon. This is the main process observed in SPDC.
- **Multi-Photon Pair Generation.** Very rarely the higher-order terms in the expansion produce multi-photon pairs, such as two signal photons and two idler photons, or even more. While possible, these events are much less probable and so have limited impact on single entangled photon pair creation.

For this analysis, the focus is on the single photon pair generation, where the output consists of a signal photon entangled with an idler photon.

The single pair component of the wave function includes terms related to the momentum and polarization states of the signal and idler photons, shown as ket notations $|k',s'\rangle$. The expression also contains a phase matching term, represented by a sinc function sin(ΔkL)/Δk, ensuring conservation of momentum and phase coherence. In the time limit where t≫1/ω, the sinc function becomes a Dirac delta function, enforcing energy conservation across any signal photons and idler photons. Further examination of the wave function reveals dependencies on both polarization and momentum for the generated photon pairs. For example, terms like $\epsilon_{k's'}$ $\epsilon^*_{k's'}$ show the polarization components of the signal and idler photons, indicating that the polarization states are intrinsically entangled. Additionally, the wave function's form indicates that the full wave function is actually a squeezed state.

How can we effectively visualize what spontaneous parametric down conversion looks like?

Visualizing parts of these equations can make this quantitative explanation a bit more concrete. The functional form for the wave function for down-converted photons will have a component from the pump (V) and a component from phase-matching (Φ).



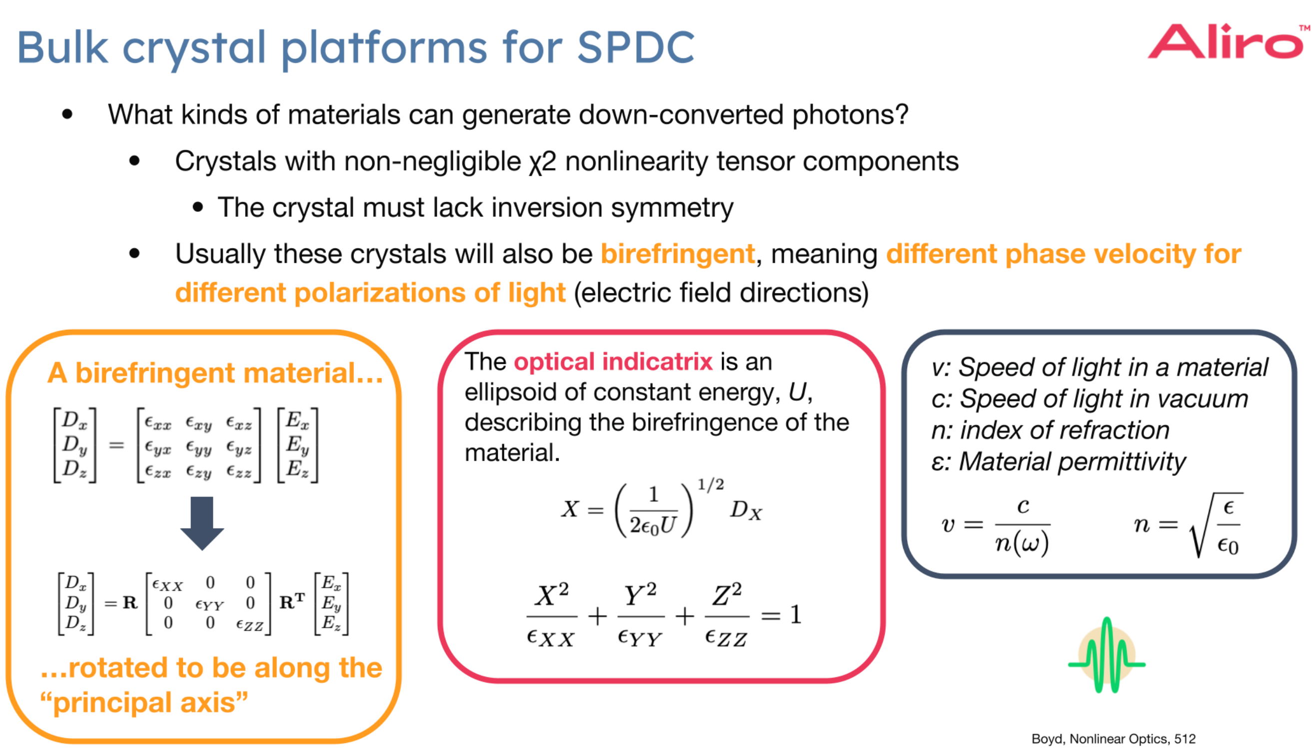
Note that this expression above does not include the polarization dependencies, and instead the quantum states are written only with respect to the momentum of the signal photon and momentum of the idler photon, specifically the transverse momenta which are written as **q**. Within this expression, there are two key elements: the pump beam function and the phase matching function. By visualizing these elements in momentum space, it's possible to gain insight into their structure and behavior. The pump beam is assumed to be a Gaussian beam with a complex phase, though it doesn't necessarily have to be Gaussian—this choice simply provides a clear example. When plotted in momentum space, we see the Gaussian pump beam has a radially symmetric Gaussian shape and a complex phase. The phase matching function for this particular example, shown in the image for a Type-I SPDC process, is a radially symmetric ring in momentum space and contains a sinc term, which is essential for conserving momentum across the process. Specifics of Type-I SPDC will be discussed later in this paper, but this visualization shows what phase matching looks like.

Bulk crystal platforms for SPDC

What kinds of materials can actually generate these down converted photons?

Nonlinear bulk crystals are a popular platform: they have non-negligible $\chi^{(2)}$ nonlinearity tensor components. The presence of $\chi^{(2)}$ in the interaction Hamiltonian governs the nonlinear interaction between photons, enabling the conversion of a single high-energy pump photon into two lower-energy, entangled photons. For a crystal to exhibit this nonlinearity, it has to lack inversion symmetry in its crystal structure.

Many crystals used for SPDC are birefringent, meaning light traveling through the crystal will have different phase velocities for different polarization of the light (directions of the oscillating electric field). This birefringence (associated with the linear susceptibility, ($\chi^{(1)}$) is not directly part of the nonlinear interaction but is useful because it allows for phase matching in the SPDC platforms. It can also be used to categorize $\chi^{(2)}$ materials because it affects how crystals transmit light based on polarization and orientation.



There are two different polarization vectors of light, "extraordinary" and "ordinary", for a birefringent material in a given orientation and with given material properties. Light polarized along the extraordinary vector or ordinary vector (in no particular order) will either have the fastest or the slowest phase velocity when traveling through the birefringent material. How can we calculate the direction of the extraordinary and ordinary vectors, and how can we figure out which vector will correspond to a faster phase velocity?

To answer these questions, we can construct something known as the “optical indicatrix”. To construct his indicatrix, we begin with several equations. The box on the right above shows an equation for the speed of light, v, in a birefringent material that is determined by the material’s index of refraction, n, and can be expressed as: $v = c / n$ where c is the speed of light in a vacuum. The index of refraction, n, can also be written with respect to the permittivity ϵ: it will be equal to the square root of this permittivity ϵ divided by the permittivity of free space, ϵ_0. The permittivity in its full form is a tensor.

Next we can derive the displacement field D of light within a birefringent crystal. In matrix form, shown in the box on the far left, this displacement field incorporates the permittivity tensor ϵ_{ij} , such that $D=\epsilon_{ij} E$, where E is the electric field vector of the light. By rotating the crystal to be oriented along its principal axis, this permittivity matrix can be diagonalized to simplify the calculation of permittivity components. Most permittivity components go to zero, leaving three principal terms: ϵ_{xx} ,ϵ_{yy} , and ϵ_{zz}.

Next, the optical indicatrix can be defined. The optical indicatrix is an ellipsoid of constant energy U which describes the birefringence of a material. As shown in the middle box, keeping energy constant, the equation for an ellipsoid that will define this ellipsoid would be: $(x^2 / \epsilon_{xx}) + (y^2 / \epsilon_{yy}) + (z^2 / \epsilon_{zz}) = 1$, where X, Y, and Z are defined with respect to the components of the displacement field vector.

Bulk crystal platforms for SPDC

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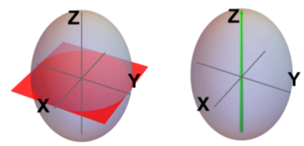
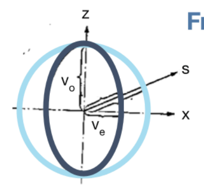
- **Birefringence** can be understood by considering a material's **optical axis**
- Optical axis:
  - "The number of diameters with the property that a plane section at right angles to them through the center of the ellipsoid is a **circle**." - Born and Wolf, *Principles of Optics*, 679.
  - "An **axis** around which linear optical properties display rotational symmetry." Boyd, *Nonlinear Optics* 43.
  - "An optic axis of a crystal is a direction in which a ray of transmitted light suffers no birefringence." (Wikipedia, *Optic axis of a crystal*)
- Ordinary polarization: phase velocity is independent of the direction of propagation.
- Extraordinary polarization: phase velocity is dependent on the angle between the normal of the wave and the optical axis.



Optical indicatrix

$$\frac{X^2}{\epsilon_{XX}} + \frac{Y^2}{\epsilon_{YY}} + \frac{Z^2}{\epsilon_{ZZ}} = 1$$

Two solutions for the velocity of a light wave in the crystal are possible: ordinary polarization and a combination of extraordinary and ordinary. Where the solutions equal each other, there is no birefringence.



Fresnel's equation of wave normals

$$\frac{s_x^2}{(v_p^2 - v_x^2)} + \frac{s_y^2}{(v_p^2 - v_y^2)} + \frac{s_z^2}{(v_p^2 - v_z^2)} = 0$$

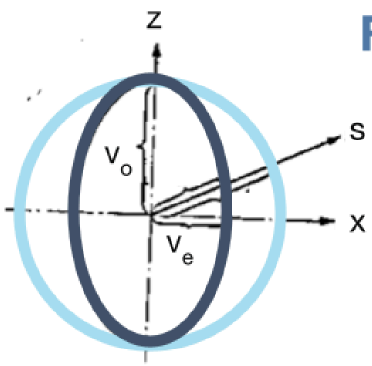
Born and Wolf, Principles of Optics, 671, 680

The optical indicatrix can allow us to define something called the "optical axis". From the optical axis, we can understand more about ordinary and extraordinary polarization, and SPDC. The optical axis can be defined in multiple ways, but all of the definitions are equivalent. The first definition of the optical axis is that it is equal to "the number of diameters with the property that a plane section at right angles to them through the center of the ellipsoid is a circle."[Born, Wolf] This ellipsoid is the optical indicatrix. In the image above, the optical indicatrix is in gray, and a red plane intersects the ellipsoid. This intersection forms a circle, and perpendicular to this red circle will be the optical axis of this material (here along the Z axis).

Another definition for the optical axis is that it's "an axis around which linear optical properties display rotational symmetry."[Boyd] In the image above, to the right of the gray ellipsoid is another gray ellipsoid with a green axis, highlighting the Z axis. If the ellipsoid is rotated around this green axis, the shape of the ellipsoid will not change: the ellipsoid is rotationally symmetric around the green axis (again, along the Z axis in the image).

The third definition of an optical axis is that it's the "direction in which a ray of transmitted light suffers no birefringence."[Wikipedia] It might be difficult to see how this definition is equivalent to these previous definitions. One way to see this equivalence is by looking at the Fresnel equation of wave normals. The Fresnel equation of wave normals, at the bottom of the image above, is an equation relating the s vector, where s is the wave normal pointing along the direction of light propagation, to the phase velocity of the light, v_p , in relation to the x, y, and z directions of the crystal material represented by the variables v_x, v_y and v_z. Note that the Fresnel equation of wave normals is in s-space (wave-normal space), while the optical indicatrix equation is in D-space (displacement vector space).

Although the two are not exactly the same equation, Fresnel's equation of wave normals does relate back to the optical indicatrix equation. The two are equivalent when considering the requirement that the s vector, the wave normal vector, dotted into the D vector, the displacement vector, equals zero, which is simply a statement that the displacement vector is orthogonal to the direction of travel of the light. Solving for the phase velocity produces two solutions. The plot at the bottom of the image above[Born, Wolf] shows a light blue circle as well as a darker blue ellipse. These are both solutions to the Fresnel equation of wave normals for the phase velocity, given different polarizations (extraordinary or ordinary). One of these solutions, the light blue circle, will be the ordinary polarization solution.



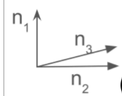
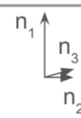
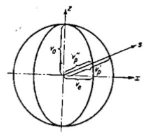
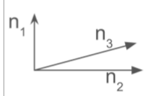
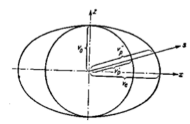
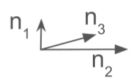
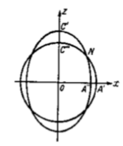
Fresnel's equation of wave normals

$$\frac{s_x^2}{(v_p^2 - v_x^2)} + \frac{s_y^2}{(v_p^2 - v_y^2)} + \frac{s_z^2}{(v_p^2 - v_z^2)} = 0$$

For this solution, the phase velocity is independent of the direction of propagation. No matter which direction s points, the resulting phase velocity for light with ordinary polarization will be the same. However, the other solution, the dark blue ellipse, is the solution for the extraordinary polarization. For this solution, phase velocity will be dependent on the angle between the normal of the wave and the optical axis. Recall that the optical axis of a crystal is the direction in which a ray of transmitted light will have the same phase velocity, no matter what its polarization is. The intersection of the two solutions for phase velocity, where the dark blue ellipse (extraordinary polarization) overlaps with the light blue circle (ordinary polarization) occurs along the optical axis of the crystal which is aligned with the z-axis, the optical axis. Along this axis, the phase velocities for the ordinary and extraordinary rays are identical, which results in no birefringence.

Classification of bulk crystal platforms for SPDC

Bulk crystals used in spontaneous parametric down-conversion (SPDC) can be classified according to their optical type. The optical type is determined by the number of optical axes in the crystal structure. This classification is helpful for understanding how different crystals interact with light in order to achieve efficient SPDC. Here, we explore the major categories of crystals based on their optical axes.

Crystal structure	Crystal optical type	Indices of refraction	Phase velocities
Cubic	Isotropic	 $(n_1 = n_2 = n_3)$	$v_p = \frac{c}{n}$
Trigonal, tetragonal, or hexagonal	**Positive** uniaxial ($n_e > n_o$; $v_e < v_o$)	 $n_1 = n_e$ $n_2 = n_3 = n_o$ $(n_1 \neq n_2 = n_3)$	
Trigonal, tetragonal, or hexagonal	**Negative** uniaxial ($n_e < n_o$; $v_e > v_o$)	 $n_1 = n_e$ $n_2 = n_3 = n_o$ $(n_1 \neq n_2 = n_3)$	
Triclinic, monoclinic, or orthorhombic	Biaxial	 $(n_1 \neq n_2 \neq n_3)$	

Born and Wolf, Principles of Optics, 679-681

Isotropic crystals. This is a crystal that has an infinite number of optical axes because the optical indicatrix is a sphere. Each index of refraction along each orthogonal direction is equal: $n_x = n_y = n_z$.
The type of crystal that can form this optical type are crystals that have cubic crystal structure. The equation for the phase velocity is very simple, because the index of refraction is the same along any direction.

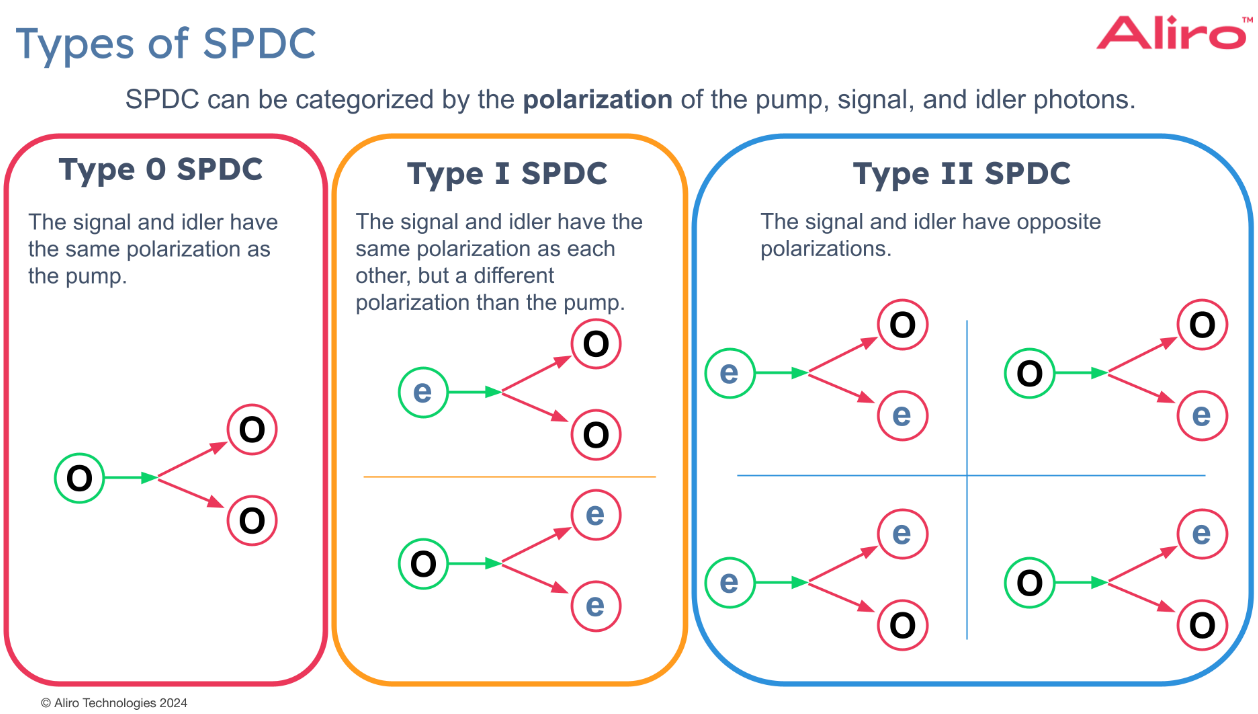
Positive uniaxial crystals. In positive uniaxial crystals, the extraordinary refractive index n_e is greater than the ordinary refractive index n_o:
$n_e > n_o$
Equivalently, this means the extraordinary velocity is less than the ordinary velocity. Types of crystals that can have this crystal optical type are trigonal, tetragonal or hexagonal crystal structures. The optical indicatrix is an ellipsoid. The index of refraction for a displacement vector along the extraordinary axis, n_e, lies on the optical axis, here the z-axis, while the index of refraction for a displacement vector along the x-axis and y-axis are the same and correspond to ordinary polarization. A plot of Fresnel's equation of wave normals will look like an inversion of the plot of the optical indicatrix in comparison. Waves propagating with the wave normal pointing along the optical axis (z axis) and the displacement vector perpendicular to the wave normal (along x or y axis) experience the ordinary phase velocity. This configuration aligns with the definition of the optical axis: along this axis, the ordinary and extraordinary refractive indices are equal, resulting in no birefringence.

Negative uniaxial crystals. In negative uniaxial crystals, the extraordinary refractive index ne is less than the ordinary refractive index n_o:
$n_e < n_o$
This means for negative uniaxial crystals, the extraordinary phase velocity is greater than the ordinary phase velocity. Crystal structures associated with this crystal optical type are again trigonal, tetragonal or hexagonal.
When plotting the solution to the Fresnel equation of wave normals for this crystal type, light traveling along the optical axis will again propagate with the ordinary phase velocity, as was the case for positive uniaxial crystals. However, if light is not traveling along the optical axis and is traveling along another axis of the crystal, the component of the light with extraordinary polarization will travel faster than the component of the light with ordinary polarization.

Biaxial crystals. Biaxial crystals have two optical axes. In these crystals, the refractive indices along each principal axis are each different and do not equal each other Perhaps counterintuitively, but easy to see graphically, this lack of symmetry results in a solution to Fresnel's equation of wave normals where there are two wave normal vectors where the ordinary and extraordinary phase velocities are equal, and hence two optical axes. The crystal structure associated with these crystals are triclinic, monoclinic, or orthorhombic.
Plotting the solution to the Fresnel equation of wave normals, the result is a circle and also an ellipse which now has even less symmetry than the previous uniaxial crystals. However, there will be two intersection axes along which the phase velocities will equal each other for the two solutions. These intersecting axes indicate the directions along which birefringence does not occur.

Polarization classification of SPDC

Another way to categorize spontaneous parametric down conversion is by the polarization of the pump, signal and idler photons.



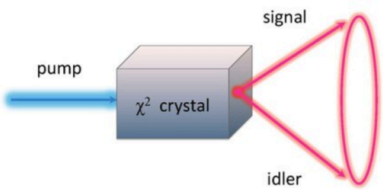
Type 0 spontaneous parametric down conversion. For this type of spontaneous parametric down conversion, the signal and idler have the same polarization as the pump. As an example, an ordinary polarized pump beam which produces an ordinary polarized signal photon and an ordinary polarized idler photon is a type 0 SPDC process.

Type I spontaneous parametric down conversion. For type I SPDC, the signal and idler photons have the same polarization as each other, but a different polarization than the pump photon. One example is the case of an extraordinary polarized pump which then produces an ordinary polarized signal photon and an ordinary polarized idler photon. For example, in a Type I SPDC process, a possible example of the polarization-entangled state generated in this process would be $|HH\rangle$, where each photon is horizontally polarized (indicated by H) as shown in the diagram below.

SPDC can be categorized by the **polarization** of the pump, signal, and idler photons.

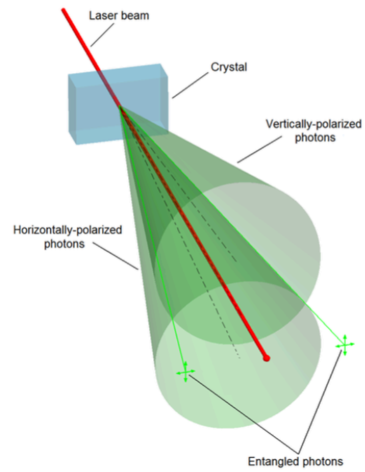
Example: Type I SPDC

- Entangled photons are emitted in pairs along a cone.
- Example of possible polarization entangled state: $|HH\rangle$



Example: Type II SPDC

- Entangled photons are emitted in pairs along two cones which intersect.
- Example of possible polarization entangled state: $\frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)$



https://en.wikipedia.org/wiki/Spontaneous_parametric_down-conversion

Type II spontaneous parametric down conversion. Type-II SPDC occurs when the signal and idler photons have opposite polarizations. In this configuration, various combinations of ordinary and extraordinary polarizations are possible for the pump, signal, and idler photons, but the distinguishing characteristic of Type-II SPDC is that the signal and idler photons must have opposite polarizations. In Type-II SPDC, the entangled photon pairs are typically emitted along two intersecting cones. By focusing on the intersection points of these cones, we can collect entangled photon pairs. A typical example of the resulting polarization-entangled state is: 1/√2 (|HV> + |VH>). Here, H and V represent horizontal and vertical polarizations, respectively. This superposition state indicates that the polarizations of the signal and idler photons are entangled, with one photon horizontally polarized and the other vertically polarized in each pair.

What determines the type of spontaneous parametric down conversion that occurs?

The type of SPDC process is determined by the $\chi^{(2)}$ nonlinearity tensor components of the nonlinear crystal which are non-zero, as well as the crystal orientation.

What determines SPDC type?

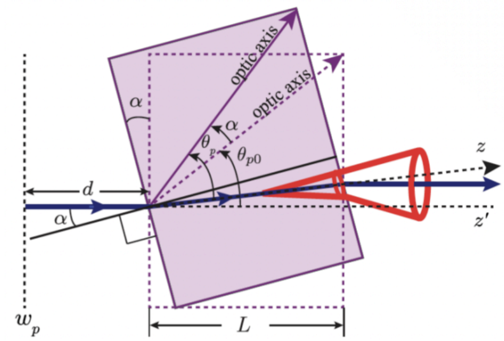
- The $\chi^{(2)}$ nonlinearity tensor components which are non-zero and the crystal orientation determine the SPDC type.
- For example, the crystal beta-barium borate (BBO) has nonzero $\chi^{(2)}_{111}$, $\chi^{(2)}_{222}$ and $\chi^{(2)}_{311}$ tensor components, allowing type 0 and type I SPDC.
  - Side note: BBO crystals are “negative uniaxial”.
- One can also achieve type II SPDC with a BBO crystal by changing the crystal orientation (tilting the crystal).
- There are different **phase-matching conditions** for different SPDC types.

Phase-matching conditions relating the k-vectors for the pump, signal, and idler for collinear phase-matching.

$$\frac{n_p(\omega_p)\omega_p}{c} = \frac{n_s(\omega_s)\omega_s}{c} + \frac{n_i(\omega_i)\omega_i}{c}$$

$$U \sim \Sigma_{ijk}\, \chi^{(2)}_{ijk} E^{(p)}_i E^{(s)}_j E^{(i)}_k$$

The $\chi^{(2)}$ should be non-zero for pump, signal, and idler of the desired polarizations.



Rotating a SPDC crystal

Karan S. et al, J. Opt. 22 (2020) 083501

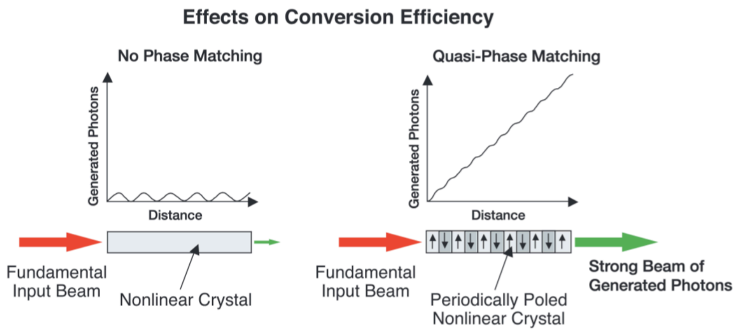
In the example above, a beta barium borate (BBO) crystal has non-zero tensor components of $\chi^{(2)}_{111}$, $\chi^{(2)}_{222}$, and $\chi^{(2)}_{311}$. This allows for Type-0 and Type-I SPDC.
BBO crystals are negative uniaxial. The crystal's birefringent properties enable specific phase-matching techniques, where the phase velocity experienced by a photon within the crystal can be adjusted by changing polarization and propagation direction. Adjusting the crystal orientation and tilting the crystal allows Type-II SPDC to also be achieved with a BBO crystal.

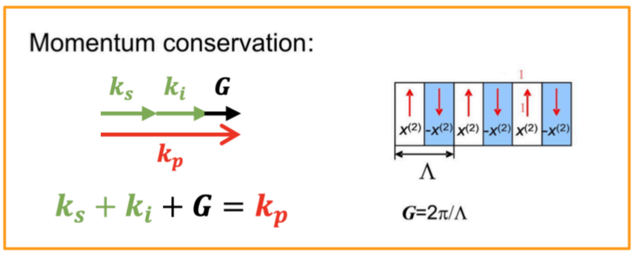
Each SPDC type requires different phase-matching conditions to satisfy momentum conservation. For example, in collinear phase matching, the phase-matching condition is a momentum conservation equation for the signal, pump and idler photons. The momentum for each photon depends on the refractive index experienced by the photon, which in turn depends on the photon’s polarization.
The SPDC type is a function of the crystal's $\chi^{(2)}$ properties, birefringence, and orientation, each of which influences phase matching and, ultimately, the efficiency and characteristics of the down-converted photon pairs produced.

Quasi-Phase Matching

Quasi-phase matching is a technique for achieving phase matching by engineering the crystal's axis orientation along its length. This is particularly useful for integrating such crystals into a waveguide structure to enable entanglement generation on an integrated photonic platform, which allows for miniaturization, and enhanced stability.

Quasi-phase matching





- Coherence length: the length over which the phase of the pump and the combined phase of signal and idler are 180 degrees from each other.
- Energy will always flow from pump to signal and idler when this phase is less than 180 degrees. Beyond 180 degrees, energy flows back from the signal to the pump frequencies.
- At each coherence length the crystal axes are flipped which allows the energy to continue to positively flow from the pump to the signal and idler frequencies.
- Allows one to use the largest nonlinear coefficient of the material in the nonlinear interaction. All the optical frequencies involved can be collinear.

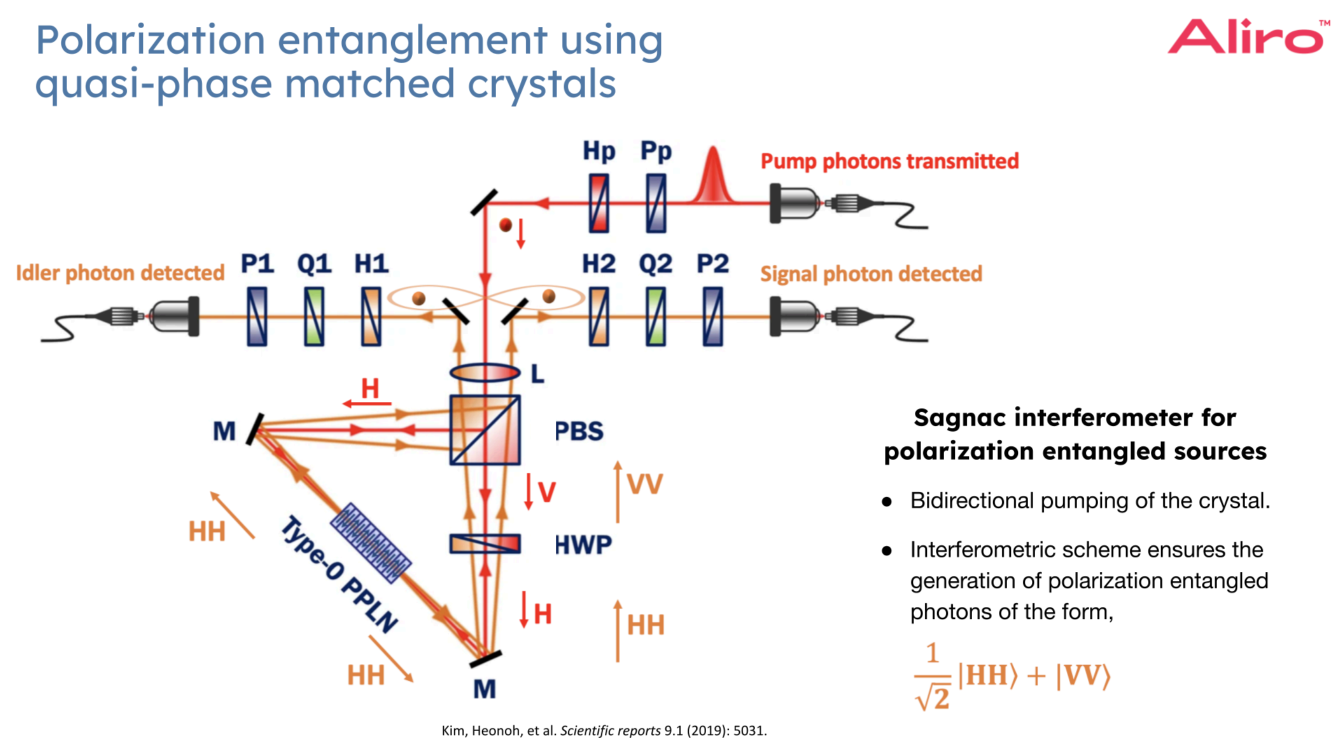
https://www.thorlabs.com/catalogpages/693.pdf

When phase matching is not achieved, the energy transfer from the pump photon to the output signal and idler photons becomes inefficient as the fields propagate through the crystal. Initially, the pump energy transfers to the output fields, but due to the phase mismatch between the input pump field and the output signal and idler fields, energy can flow back from the signal and idler photons to the pump. As this phase mismatch accumulates and reaches 180 degrees, the energy flow reverses entirely, returning from the signal and idler photons to the pump. The distance over which this 180-degree phase difference accumulates is referred to as the coherence length.

This is where quasi-phase matching provides a solution. In quasi-phase matching, the crystal's axis is inverted once every coherence length, preventing energy transfer from the output electric fields back to the pump and ensuring continuous energy transfer from the pump field to the signal and idler fields. This alteration modifies the expression for momentum conservation in quasi-phase matching: while the wave vectors corresponding to the pump, signal, and idler are present, an additional component arises due to the periodic inversion of the crystal axis. This period, referred to as the poling period, is equal to twice the coherence length of the crystal. As a result, energy continuously flows from the pump to the output fields, maximizing the efficiency of the SPDC process.

How is this useful? The quasi-phase-matching scheme enables the selection of the crystal's largest nonlinear coefficient, maximizing the efficiency of generating entangled photons with high conversion efficiency. In addition, it can be used to ensure all the optical fields are co-linearly propagating, which makes the alignment easier in the system.

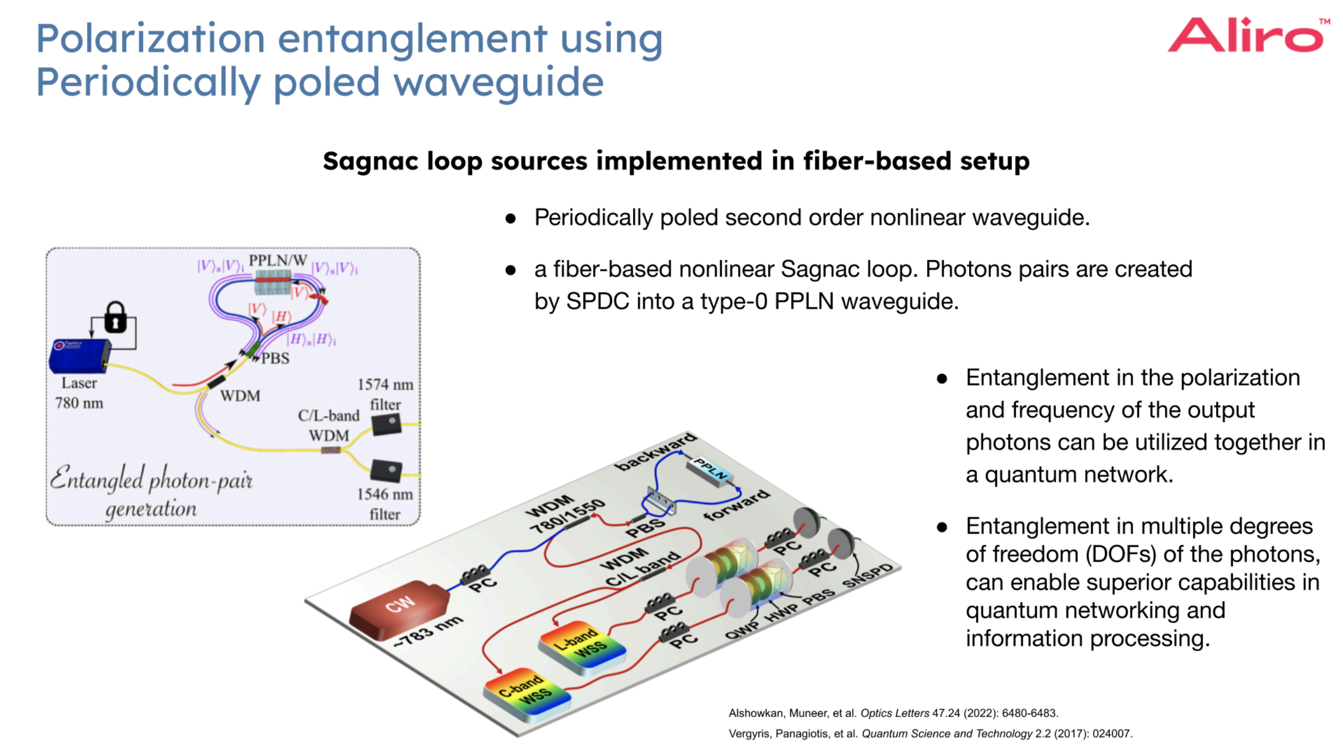
We will now explore demonstrations that enable polarization entanglement generation using quasi-phase matching, highlighting how this approach can support an alignment-free entanglement source suitable for an all-fiber setup and for hyperentanglement generation. Consider Type-0 phase matching where the input pump field shares the same polarization as the generated signal and idler fields. While this process alone does not produce polarization entanglement, a specialized setup can be designed to achieve polarization entanglement based on this configuration.



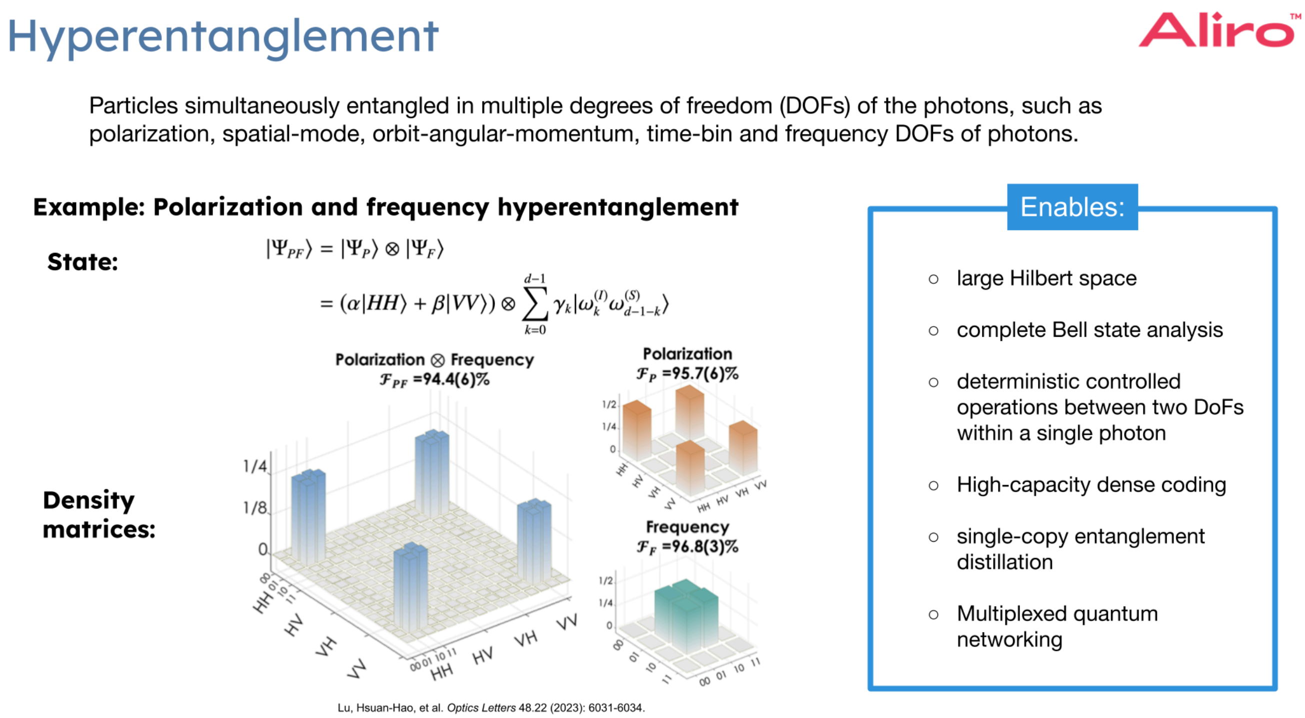
In the example above, a Sagnac loop is used in order to achieve polarization entanglement using a Type 0 quasi-phase matched source.[Kim, Heonoh, et al.] Let's consider the pump photon marked in red, that is routed to a polarizing beam splitter. The pump photon is aligned in a polarization that's either anti-diagonal or diagonal, so that the photon is present in equal superposition at both output ports of the polarizing beam splitter. One of the output ports will have a polarization of H, the other will have polarization of V.

After the beam splitter, one of these outputs passes through a half-wave plate that will rotate the V polarization to H. This setup enables bidirectional pumping of a Type-0, periodically poled crystal, with pump photon contributions to the entangled state from both directions to interfere back at the polarizing beam splitter. These output contributions of signal and idler generated will both have horizontal polarization. The output fields are represented in the orange color in the figure above. In one of the paths, the horizontal polarization is converted to vertical polarization, and both output fields in both directions are going to then traverse back towards the polarizing beam splitter, resulting in an entangled output state of the form 1/√2 (|HH> + |VV>).

A key advantage of using quasi-phase-matching is that this setup can be integrated onto an on-chip platform.[Vergyris, Panagiotis, et al.] For example, a waveguide with a nonlinear $\chi^{(2)}$ susceptibility of significant amplitude is periodically poled along its length to achieve efficient conversion from the input pump photon to the output signal and idler photons. Combined with fiber-optic coupling, this scheme provides a stable source, particularly useful when implementing entangled photon sources for deployment in fiber-optic networks. Examples of this are in the image below.



In addition to this integration, it is also possible to harness additional degrees of freedom that are there in the generated photons.[Alshowkan, Muneer, et al.] Due to the conservation of energy and momentum, these photons can exhibit entanglement across several degrees of freedom simultaneously, such as frequency, polarization, spatial mode, angular momentum, and more, depending on the specific SPDC process used. It is possible to ensure simultaneous entanglement across multiple degrees of freedom, enhancing the versatility and capability of the entangled photon pairs.



For a quantitative view, consider the state of these two photons as a tensor product of the states across different degrees of freedom, such as polarization and frequency.[Lu, Hsuan-Hao, et al.] This increases the Hilbert space available for quantum operations, enabling enhanced capabilities such as deterministic controlled operations between two degrees of freedom, higher capacity in dense coding protocols, and single-copy entanglement distillation. This multi-degree-of-freedom-entanglement is also advantageous for multiplexing in quantum networks.

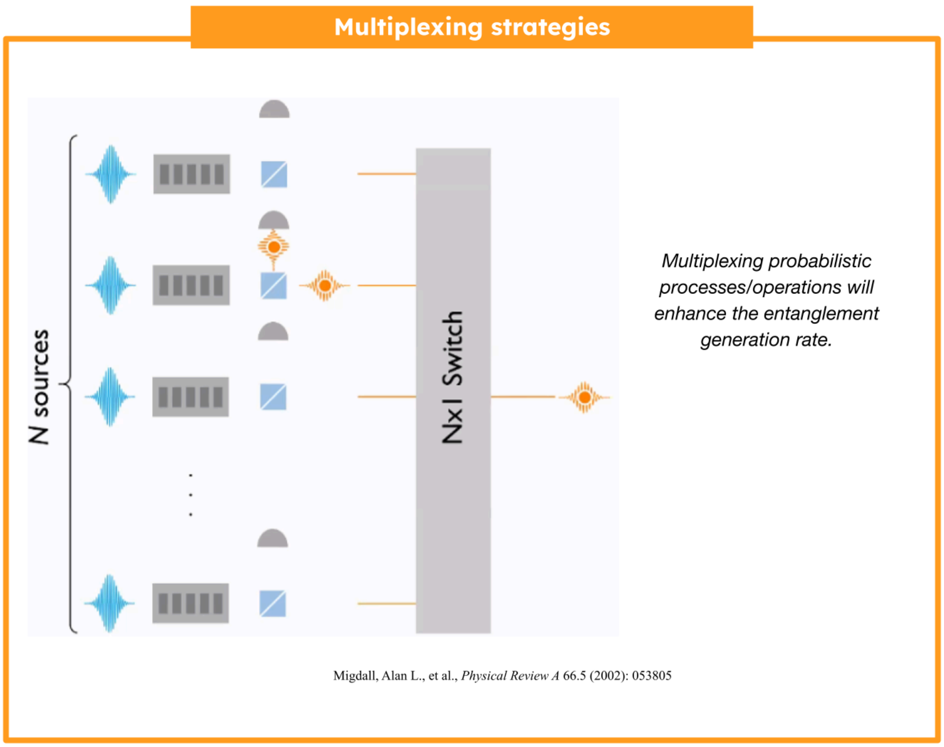
Further Useful Directions with Entangled Sources

Multiplexing

There are interesting directions that we can take to realize a scalable deployment of these entangled sources in a practical quantum network. One such direction would be multiplexing.[Migdall, Alan L., et al.] There is a fairly low probability of generating entangled photons when the pump photon is seeded into the nonlinear medium. It is essential for us to achieve high entanglement generation rates in order to scale distances in a practical fiber optic quantum network, and also scale the number of nodes in the system.

Multiplexing addresses this by using multiple sources simultaneously, thereby increasing the overall probability of successful entanglement generation. This can involve:

- Using multiple sources in parallel, which collectively enhance the generation probability.
- Utilizing multiple frequency bands to create entangled photons across various wavelengths.
- Time-multiplexing across different time windows to increase the likelihood of successful entanglement events.

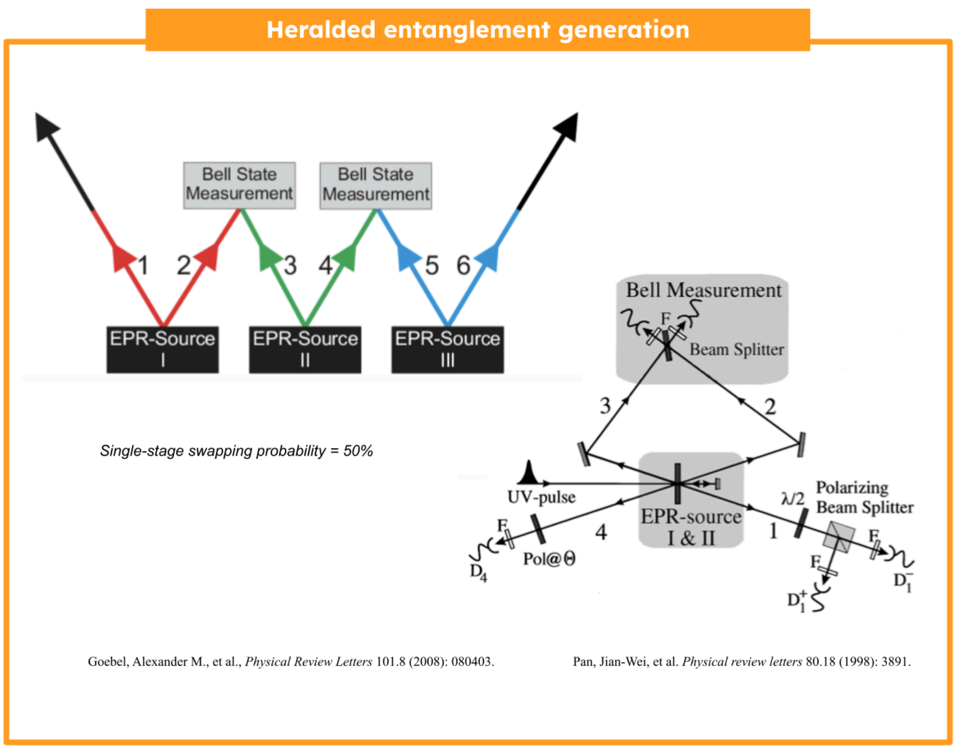


These schemes for multiplexing can enhance the entanglement generation rate further.

Heralded Entanglement Generation[Goebel, Alexander M., et al., Pan, Jian-Wei, et al.]

For long-distance entanglement distribution between distant nodes, heralded entanglement generation is another critical technique. In this approach, multiple entanglement sources are placed in the network. At an intermediate node, photons from two different sources are brought together, where a Bell state measurement is performed, projecting these photons onto a maximally entangled state, also known as a Bell state.

This Bell state measurement effectively ensures that two other photons become entangled, even though they never directly interact. By cascading this system of heralded entanglement generation across multiple nodes, it's possible to achieve entanglement between two distant photons (e.g., photons one and six in the image above), supporting long-distance entanglement without requiring direct interaction between these distant photons.



This cascaded system of heralded entanglement generation is useful in creating scalable and practical quantum networks, making it possible to establish robust entanglement across extended distances.

The Future is Entanglement-based Quantum Networks

As we look to the future of secure communications, entanglement-based quantum networks are at the forefront. Building these networks requires sophisticated software and hardware, such as entangled photon sources.

Entanglement-based quantum networks are being built today by a variety of organizations for a variety of use cases – benefiting organizations internally, as well as providing great value to an organization's customers. Telecommunications companies, national research labs, and systems integrators are just a few examples of the organizations Aliro is helping to leverage the capabilities of quantum secure communications.

Building entanglement-based quantum networks that use entanglement is no easy task. It requires:

- Emerging hardware components necessary to build the network.
- The software necessary to design, simulate, run, and manage the network.
- A team with expertise in the fundamental science of entanglement-based quantum networks and classical networking.
- Years of hard work and development.

This may seem overwhelming, but Aliro is uniquely positioned to help you build your quantum network. The steps you can take to ensure your organization is meeting the challenges and leveraging the benefits of the quantum revolution are part of a clear, unified solution already at work in networks like the EPB Quantum Network℠ powered by Qubitekk in Chattanooga, Tennessee.

AliroNet™, the world's first full-stack entanglement-based network solution, consists of the software and services necessary to ensure customers will fully meet their advanced secure networking goals. Each component within AliroNet™ is built from the ground up to be compatible and optimal with entanglement-based networks of any scale and architecture. AliroNet™ is used to simulate, design, run, and manage quantum networks as well as test, verify, and optimize quantum hardware for network performance. AliroNet™ leverages the expertise of Aliro personnel in order to ensure that customers get the most value out of the software and their investment.

Depending on where customers are in their quantum networking journeys, AliroNet™ is available in three modes that create a clear path toward building full-scale entanglement-based secure networks: (1) Emulation Mode, for emulating, designing, and validating entanglement-based quantum networks, (2) Pilot Mode for implementing a small-scale entanglement-based quantum network testbed, and (3) Deployment Mode for scaling entanglement-based quantum networks and integrating end-to-end applications. AliroNet™ has been developed by a team of world-class experts.

To get started on your Quantum Networking journey, reach out to the Aliro team for additional information on how AliroNet™ can enable secure communications.

www.alirotech.com

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